

AN ALGEBRO-GEOMETRIC STUDY OF THE UNIT ARGUMENTS ${}_3F_2(1, 1, q; a, b; 1)$, I

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ABSTRACT. Let a, b, q be rational numbers such that none of $a, b, q, q - a, q - b, q - a - b$ is an integer. Then we prove, ${}_3F_2(1, 1, q; a, b; 1)$ is a \mathbb{Q} -linear combination of log of algebraic numbers if $\{sq\} + \{s(a - q)\} + \{s(b - q)\} + \{s(q - a - b)\} = 2$ for all integers s prime to the denominators of a, b and q where $\{x\}$ denotes the fractional part of x .

1. INTRODUCTION

In this paper we study the unit argument

$$B(a, b) {}_3F_2 \left(\begin{matrix} a, b, q \\ a + b, q + 1 \end{matrix}; 1 \right) = \frac{q}{ab} \cdot {}_3F_2 \left(\begin{matrix} 1, 1, a + b - q \\ a + 1, b + 1 \end{matrix}; 1 \right) \quad (1.1)$$

for non-integers $a, b, q \in \mathbb{Q}$ (the equality follows from Dixon's formula, [5] Ch. III, 3.2 (1)). We first note that their contiguous relations for a, b and q hold (e.g. [4] §7.3), so that the complex numbers (1.1) are “essentially” same if we replace a with $a + n$ etc.

There is a very classical formula of Watson [9] (see also [5] Ex.9, p.98) which says

$$2B(a, b) {}_3F_2 \left(\begin{matrix} a, b, \frac{a+b-1}{2} \\ a + b, \frac{a+b+1}{2} \end{matrix}; 1 \right) = \psi \left(\frac{a+1}{2} \right) + \psi \left(\frac{b+1}{2} \right) - \psi \left(\frac{a}{2} \right) - \psi \left(\frac{b}{2} \right)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. The special values $\psi(\alpha)$ for $\alpha \in \mathbb{Q}$ are known by Gauss ([7] 1.7.3, p.18–19). Thus the formula of Watson implies that the unit argument (1.1) is a \mathbb{Q} -linear combination of finitely many $\log(\alpha)$'s with $\alpha \in \mathbb{Q}(\mu_\infty)$ when $2q \equiv a + b \pmod{\mathbb{Z}}$.

On the other hand, the recent work [4] by the first and second authors shows that the left hand side of (1.1) appears as *Beilinson's regulator* on the motivic cohomology groups of fibrations with multiplication. The third author pointed out that it also appears from the Fermat surfaces. Under a certain geometric assumption it is known that the regulator is written by logarithmic function. Thus one can obtain a sufficient condition for that (1.1) are written by logarithmic function. It is remarkable that our geometrical method covers not only the case of Watson's

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formula but new cases. For example it allows to show

$$2\pi \cdot {}_3F_2\left(\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{4} \\ 1, \frac{5}{4} \end{matrix}; 1\right) = \frac{12^{\frac{3}{4}}}{2} \log\left(\frac{3^{\frac{5}{4}} - 3^{\frac{3}{4}} + \sqrt{2}}{3^{\frac{5}{4}} - 3^{\frac{3}{4}} - \sqrt{2}}\right) - 12^{\frac{3}{4}} \text{Cos}^{-1}\left(\frac{3^{\frac{5}{4}} + 3^{\frac{3}{4}}}{2\sqrt{5 + 3\sqrt{3}}}\right).$$

This is distinguished from Watson's formula also in the sense that non-cyclotomic numbers appear inside of log.

This paper is organized as follows. The main theorem is stated in §2. We give two proofs of the main theorem in §3 and §4. The first proof uses the motivic cohomology of fibrations and it is due to the first and second authors. The second one uses the motivic cohomology of the Fermat surfaces, and it is due to the third author. In §5 open questions are discussed. In particular, we give a conjecture on a necessary and sufficient condition for that (1.1) is written by logarithmic function (Conjecture 5.2). This is quite likely true from the viewpoint of Beilinson's regulator, though we have no idea how to prove it.

2. MAIN THEOREM

Let $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ be the completion, and $\hat{\mathbb{Z}}^\times = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times$ the group of units. The ring $\hat{\mathbb{Z}}$ acts on the additive group \mathbb{Q}/\mathbb{Z} in a natural way, and then it induces $\hat{\mathbb{Z}}^\times \cong \text{Aut}(\mathbb{Q}/\mathbb{Z})$. We denote by $\{x\} := x - \lfloor x \rfloor$ the fractional part of $x \in \mathbb{Q}$. The map $\{-\} : \mathbb{Q} \rightarrow [0, 1)$ factors through \mathbb{Q}/\mathbb{Z} , which we denote by the same notation.

Theorem 2.1. *Let $a, b, q \in \mathbb{Q}$ be non-integers such that none of $q-a, q-b, q-a-b$ are integers. Thinking of q, a, b being elements of \mathbb{Q}/\mathbb{Z} , we assume that*

$$\{sq\} + \{s(-q+a)\} + \{s(-q+b)\} + \{s(q-a-b)\} = 2 \quad (2.1)$$

holds for all $s \in \hat{\mathbb{Z}}^\times$. Then

$$B(a, b) {}_3F_2\left(\begin{matrix} a, b, q \\ a+b, q+1 \end{matrix}; 1\right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times. \quad (2.2)$$

Here the right hand side denotes the $\overline{\mathbb{Q}}$ -linear subspace of \mathbb{C} generated by 1 and $\log \alpha$'s, $\alpha \in \overline{\mathbb{Q}}^\times$ (including the case $\log(1) = 2\pi i$).

We note that the action of $\hat{\mathbb{Z}}$ on the subgroup $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ factors through the finite quotient $(\mathbb{Z}/N\mathbb{Z})^\times$. Therefore if $a, b, q \in \frac{1}{N}\mathbb{Z}$, then it is enough that s in (2.1) runs over the set of integers k such that $\gcd(k, N) = 1$ and $0 < k < N$.

3. PROOF OF MAIN THEOREM : FIBRATION WITH RELATIVE MULTIPLICATION

The key ingredient of the proof is the *regulator formula* in [4], which we recall here. Let

$$f : X \longrightarrow \mathbb{P}^1$$

be a fibration equipped with a relative multiplication on $R^1 f_* \mathbb{Q}$ by a number field K which satisfies the following conditions. We fix a coordinate $t \in \mathbb{A}^1 \subset \mathbb{P}^1$.

- (a) The rank of the multiplication is two, i.e. $\dim_K R^1 f_* \mathbb{Q} = 2$.
- (b) f is smooth over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- (c) The local monodromy $T = T_1$ on $H_B^1(X_t, \mathbb{Q})$ at $t = 1$ is maximally unipotent, i.e. the rank of $N := \log(T)$ is $\frac{1}{2} \dim_{\mathbb{Q}} H_B^1(X)$.

Let $\chi : K \hookrightarrow \mathbb{C}$ be an embedding, and let $R^1 f_* \mathbb{C}^\chi$ denotes the χ -part which is defined to be the subspace on which $g \in K$ acts by multiplication by $\chi(g)$. Let T_p be the local monodromy (in counter-clockwise direction) on $R^1 f_* \mathbb{Q}$ at $p = 0, 1, \infty$. Let $(e^{2\pi i \alpha_1^\chi}, e^{2\pi i \alpha_2^\chi})$ (resp. $(e^{2\pi i \beta_1^\chi}, e^{2\pi i \beta_2^\chi})$) be eigenvalues of T_0 (resp. T_∞). Note that α_i^χ and β_j^χ are rationals.

Let $l \geq 1$ be an integer and $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ the map given by $\pi(t) = t^l$. Put $G^{(l)} := \text{Aut}(\pi) \cong \mu_l$. We consider a variation of Hodge structure $\mathcal{M}^{(l)} := \pi_* \mathbb{Q} \otimes R^1 f_* \mathbb{Q}$ and cohomology groups

$$H^{(l)} := H^1(\mathbb{P}^1, j_* \mathcal{M}^{(l)}), \quad M^{(l)} := H^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \mathcal{M}^{(l)}). \quad (3.1)$$

Put

$$E := \bigoplus_{p=0,1,\infty} E_p, \quad E_p := (R^1 j_* \mathcal{M}^{(l)})_p = \text{Coker}[T_p - 1 : \mathcal{M}^{(l)} \rightarrow \mathcal{M}^{(l)}]. \quad (3.2)$$

There is an exact sequence

$$0 \longrightarrow H^{(l)} \longrightarrow M^{(l)} \longrightarrow E \longrightarrow 0 \quad (3.3)$$

of mixed de Rham-Hodge structures and all terms are equipped with the multiplication by the ring $K[G^{(l)}]$.

For $k \in (\mathbb{Z}/l\mathbb{Z})^\times$ let $\varepsilon_k : G^{(l)} \cong \mu_l \rightarrow \mathbb{C}$ be a group homomorphism given by $\zeta \mapsto \zeta^k$. Then $\varepsilon_k \otimes \chi$ defines a homomorphism $K[G^{(l)}] \rightarrow \mathbb{C}$ of \mathbb{Q} -algebra in a natural way. Let L be the coimage of $\varepsilon_k \otimes \chi$. Let $e \in K[G^{(l)}]$ be the idempotent associated to L , i.e. $e^2 = e$ and $eK[G^{(l)}] = L$.

Suppose that none of rational numbers

$$\frac{k}{l} + \alpha_1^\chi, \quad \frac{k}{l} + \alpha_2^\chi, \quad -\frac{k}{l} + \beta_1^\chi, \quad -\frac{k}{l} + \beta_2^\chi$$

belongs to \mathbb{Z} . It is not hard to show that $eE_0 = eE_\infty = 0$, $\dim_L eH^{(l)} = \dim_L eE = 1$ and eE has the Hodge type $(2, 2)$. Then taking the $(\varepsilon_{-k} \otimes \overline{\chi})$ -part of (3.3), one has the connecting homomorphism

$$\rho^{\varepsilon_{-k} \otimes \overline{\chi}} : \overline{\mathbb{Q}} \longrightarrow \mathbb{C} / \text{Per}([eH^{(l)}]^{\varepsilon_{-k} \otimes \overline{\chi}}) + \overline{\mathbb{Q}}$$

where $\text{Per}([eH^{(l)}]^{\varepsilon_{-k} \otimes \overline{\chi}})$ denotes the period of $(\varepsilon_{-k} \otimes \overline{\chi})$ -part in the sense of [6] (see [4] §4.2 for detail).

Theorem 3.1 (Regulator formula, [4] Theorem 4.3). *Let the notation and assumption be as above. Then $\rho^{\varepsilon_{-k} \otimes \overline{\chi}}(1)$ is a $\overline{\mathbb{Q}}$ -linear combination of*

$$1, \quad c \cdot \Gamma \left(\frac{\alpha_1^\chi + k/l, \alpha_2^\chi + k/l}{k/l - \beta_1^\chi, k/l - \beta_2^\chi} \right), \quad (3.4)$$

for some $c \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times$ and

$$B(\alpha_1^\chi + \beta_1^\chi, \alpha_1^\chi + \beta_2^\chi) {}_3F_2 \left(\frac{\alpha_1^\chi + \beta_1^\chi, \alpha_1^\chi + \beta_2^\chi, \alpha_1^\chi + k/l}{2\alpha_1^\chi + \beta_1^\chi + \beta_2^\chi, \alpha_1^\chi + k/l + 1; 1} \right). \quad (3.5)$$

In addition the coefficient of (3.5) is non-zero.

The connecting homomorphism $\rho^{\varepsilon-k\otimes\overline{\chi}}$ is *Beilinson's regulator map* on the motivic cohomology group. Let us explain it more precisely. Let

$$\begin{array}{ccccc} X^{(l)} & \xrightarrow{i} & X_l & \longrightarrow & X \\ & \searrow f^{(l)} & \downarrow & \square & \downarrow f \\ & & \mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^1 \end{array}$$

where i is the desingularization. Put $D^{(l)} := f^{(l)-1}\pi^{-1}(1)$ a union of l copies of the fiber $f^{-1}(1)$. There are canonical isomorphisms $H_1(D^{(l)}, \mathbb{Q}(-2)) \cong E_1$ and

$$H^{(l)} \cong \text{Ker}[H^2(X^{(l)}, \mathbb{Q})/\langle \text{fib.div.} \rangle \rightarrow H^2(f^{(l)-1}(t), \mathbb{Q})]$$

where “fib.div.” denotes the fibral divisors for $f^{(l)}$. Then the regulator maps induces a commutative diagram

$$\begin{array}{ccc} H^3_{\mathcal{M}, D^{(l)}}(X^{(l)}, \mathbb{Q}(2)) & \xrightarrow{\quad} & H^3_{\mathcal{M}}(X^{(l)}, \mathbb{Q}(2)) \\ \text{reg}_{D^{(l)}} \downarrow & & \downarrow \text{reg}_{X^{(l)}} \\ H_1(D^{(l)}, \mathbb{Q}) & \xrightarrow{\rho} \text{Ext}^1(\mathbb{Q}, eH^{(l)}(2)) \xrightarrow{\subset} & \text{Ext}^1(\mathbb{Q}, H^2(X^{(l)}, \mathbb{Q}(2))/\langle \text{fib.div.} \rangle) \end{array}$$

where ρ is the connecting homomorphism to the Yoneda extension group of mixed de Rham-Hodge structures ([4] Proposition 4.4).

Lemma 3.2. *If $eH^{(l)}$ is a Hodge structure of type $(1, 1)$, then $\rho^{\varepsilon-k\otimes\overline{\chi}}(1) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\log\overline{\mathbb{Q}}^\times$, and hence so is (3.5) by the regulator formula.*

We note that the assumption in Lemma 3.2 implies that the second term (3.4)

$$\Gamma \left(\begin{array}{c} \alpha_1^\chi + k/l, \alpha_2^\chi + k/l \\ k/l - \beta_1^\chi, k/l - \beta_2^\chi \end{array} \right) \in \overline{\mathbb{Q}}^\times$$

by the Lemma of Koblitz-Ogus ([6]).

Proof. Note that there is the canonical isomorphism $\text{Ext}^1(\mathbb{Q}, eH^{(l)}(2)) \cong \mathbb{C}/\mathbb{Q}(1)^\oplus$ as $eH^{(l)} \cong \mathbb{Q}(-1)^\oplus$. Then $\rho^{\varepsilon-k\otimes\overline{\chi}}$ can be regarded as the $(\varepsilon_{-k} \otimes \overline{\chi})$ -part of ρ (cf. [4], 4.2), and hence it is enough to see that the image of ρ lies in the subspace $(\overline{\mathbb{Q}} + \overline{\mathbb{Q}}\log\overline{\mathbb{Q}}^\times)^\oplus \subset \mathbb{C}/\mathbb{Q}(1)^\oplus$. Since reg_D is surjective, it is enough to see that so does the image of $H^3_{\mathcal{M}, D^{(l)}}(X^{(l)}, \mathbb{Q}(2))$.

Let $N^r(X^{(l)}) = N_{\dim X - r}(X^{(l)}) \subset H^{2r}(X^{(l)}, \mathbb{Q})$ be the subspace generated by algebraic cycles of codimension r . Note that they are generated by cycles defined over $\overline{\mathbb{Q}}$. By the assumption $eH^{(l)} \subset N^1(X^{(l)})$. The pairing $N^1(X^{(l)}) \otimes N_1(X^{(l)}) \rightarrow \mathbb{Q}$ is non-degenerate by the non-degeneracy of the pairing on the Néron-Severi groups. This implies that there is a (not necessarily connected) smooth projective curve C and a morphism $C \rightarrow X^{(l)}$ such that the image of C intersects with $D^{(l)}$ properly and the composition $eH^{(l)} \rightarrow H^2(X^{(l)})/\text{fib.div} \rightarrow H^2(C)$ is injective. Then the assertion follows from the following commutative diagram

$$\begin{array}{ccc} H^3_{\mathcal{M}, D^{(l)}}(X^{(l)}, \mathbb{Q}(2)) & \xrightarrow{\quad} & H^3_{\mathcal{M}, D^{(l)} \cap C}(C, \mathbb{Q}(2)) \cong (\overline{\mathbb{Q}}^\times)^\oplus \\ \rho \text{reg}_{D^{(l)}} \downarrow & & \downarrow \log \\ \text{Ext}^1(\mathbb{Q}, eH^{(l)}(2)) & \xrightarrow{\quad} & \text{Ext}^1(\mathbb{Q}, H^2(C, \mathbb{Q}(2))) \cong \mathbb{C}/\mathbb{Q}(1)^\oplus. \end{array}$$

□

Lemma 3.3. *Let the notation and assumption be as in Theorem 3.1. Then $eH^{(l)}$ is a Hodge structure of type $(1, 1)$ if and only if*

$$\left\{ s \left(\frac{k}{l} + \alpha_1^X \right) \right\} + \left\{ s \left(\frac{k}{l} + \alpha_2^X \right) \right\} + \left\{ s \left(-\frac{k}{l} + \beta_1^X \right) \right\} + \left\{ s \left(-\frac{k}{l} + \beta_2^X \right) \right\} = 2$$

for all $s \in \hat{\mathbb{Z}}^\times$. In this case we have

$$B(\alpha_1^X + \beta_1^X, \alpha_1^X + \beta_2^X) {}_3F_2 \left(\begin{matrix} \alpha_1^X + \beta_1^X, \alpha_1^X + \beta_2^X, \alpha_1^X + k/l \\ 2\alpha_1^X + \beta_1^X + \beta_2^X, \alpha_1^X + k/l + 1 \end{matrix}; 1 \right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times.$$

Proof. The former is a consequence of an explicit formula of the Hodge type of $eH^{(l)}$ which is proven by the Riemann-Roch-Hirzebruch theorem ([2] §5.5, see also [4] Lemma 3.1 and Remark 4.2). The latter follows from this and Lemma 3.2. □

To prove Theorem 2.1 it is enough to find a fibration f and an embedding χ such that

$$a = \alpha_1^X + \beta_1^X, \quad b = \alpha_1^X + \beta_2^X, \quad q = \alpha_1^X + \frac{k}{l}.$$

We do not need to take q into account because k/l can be an arbitrary rational number. Thus the following lemma finishes the proof of Theorem 2.1.

Lemma 3.4. *Let $a, b \in \mathbb{Q}$ be non-integers. Then there is a fibration f and an embedding χ such that $a \equiv \alpha_1^X + \beta_1^X$ and $b \equiv \alpha_1^X + \beta_2^X$ modulo \mathbb{Z} .*

Proof. Let $N > 1$ be an integer. Let $0 < i, j < N$, $0 \leq k < N$ be arbitrary integers such that $\gcd(N, i, j) = 1$. We consider the *hypergeometric fibration* (cf. [1]):

$$g : Y \longrightarrow \mathbb{P}^1, \quad g^{-1}(t) : t^k y^N = x^i (1-x)^j (1-tx)^{N-j}.$$

There is the automorphism $\sigma \in \text{Aut}(Y)$ given by $\sigma(x, y, t) = (x, \zeta_N y, t)$ with ζ_N a primitive N th root of unity. Then $R^1 g_* \mathbb{Q}$ has a multiplication by $\mathbb{Q}[\sigma] \cong \mathbb{Q}[T]/(T^N - 1)$. Let $\text{Jac}_{Y/\mathbb{P}^1} \rightarrow \mathbb{P}^1$ be the Jacobian fibration of g . Then we take the component of it for the projection $e_N : \mathbb{Q}[\sigma] \rightarrow \mathbb{Q}[\sigma]/(\Phi(\sigma))$ where $\Phi_N(T) \in \mathbb{Q}[T]$ is the irreducible polynomial of ζ_N :

$$f : X = e_N(\text{Jac}_{Y/\mathbb{P}^1}) \longrightarrow \mathbb{P}^1.$$

This is the desired fibration. Indeed the general fiber $f^{-1}(t)$ is a $\varphi(N)$ -dimensional abelian varieties ([1] Theorem 6.7), and equipped with multiplication by $\mathbb{Q}(\zeta_N)$. The periods of $f^{-1}(t)$ are described by the Gauss hypergeometric functions and hence one has the monodromy representation as the triangulated group of hypergeometric functions. In particular one has that the local monodromy T_1 at $t = 1$ is unipotent and

$$\alpha_1^X = \frac{s(i+j+k)}{N}, \quad \alpha_2^X = \frac{sk}{N}, \quad \beta_1^X = \frac{-s(i+k)}{N}, \quad \beta_2^X = \frac{-s(j+k)}{N}$$

for some $s \in (\mathbb{Z}/N\mathbb{Z})^\times$. When $\chi : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ runs over all embeddings, s runs over all $s \in (\mathbb{Z}/N\mathbb{Z})^\times$. Since $\alpha_p^X + \beta_q^X \notin \mathbb{Z}$ for any $p, q = 1, 2$, the local monodromy T_1 cannot be trivial, and hence it is maximal unipotent. Thus all the conditions (a), (b) and (c) are satisfied. Since i, j can be arbitrary, we are done. □

4. ALTERNATIVE PROOF : FERMAT SURFACES

We give an alternative proof of the main theorem 2.1.

We begin with the integral formula of the hypergeometric functions:

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}; z \right) \\ &= C \int_0^1 \int_0^1 t_1^{\alpha_1-1} t_2^{\alpha_2-1} (1-t_1)^{\beta_1-\alpha_1-1} (1-t_2)^{\beta_2-\alpha_2-1} (1-zt_1t_2)^{-\alpha_3} dt_1 dt_2, \end{aligned}$$

$$C = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2-\alpha_2)}.$$

Set $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2) = (a, q, b, a+b, q+1)$. Changing the variables $x = t_1$ and $y = (1-t_1)/(1-zt_1t_2)$, one has

$$z^q t_1^{a-1} t_2^{q-1} (1-t_1)^{b-1} (1-zt_1t_2)^{-b} dt_1 dt_2 = x^{a-q-1} y^{b-q-1} (x+y-1)^{q-1} dx dy$$

and hence

$$B(a, b) {}_3F_2 \left(\begin{matrix} a, q, b \\ a+b, q+1 \end{matrix}; z \right) = qz^{-q} \int_{E_z} x^{a-q-1} y^{b-q-1} (x+y-1)^{q-1} dx dy \quad (4.1)$$

where E_z is the domain in xy -plane corresponding to $0 \leq t_1, t_2 \leq 1$. Suppose $a, b, q \in \frac{1}{N}\mathbb{Z}$. We take new variables u, v, w such that

$$u^N = x, \quad v^N = y, \quad w^N = u^N + v^N - 1 = x + y - 1.$$

Then (4.1) is written as

$$B(a, b) {}_3F_2 \left(\begin{matrix} a, q, b \\ a+b, q+1 \end{matrix}; z \right) = N^2 q z^{-q} \int_{\Delta_z} u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} du dv \quad (4.2)$$

where Δ_z is an arbitrary domain in uv -plane which corresponds to E_z . Substitute $z = 1$. We choose a domain

$$\Delta = \Delta_1 = \{(u, v) \in \mathbb{R}^2 \mid 0 \leq u, v \leq 1, 1 \leq u^N + v^N\} \quad (4.3)$$

and then (4.2) becomes

$$B(a, b) {}_3F_2 \left(\begin{matrix} a, q, b \\ a+b, q+1 \end{matrix}; 1 \right) = N^2 q \int_{\Delta} u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} du dv. \quad (4.4)$$

We shall give a motivic interpretation of the integral in (4.4).

The differential form $\eta := u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} du dv$ defines a cohomology class in de Rham H^2 of the *Fermat surface*

$$S : u^N + v^N - 1 = w^N.$$

Let $G = \mu_N^3$ be the group which acts on S by $\sigma(u, v, w) = (\zeta_1 u, \zeta_2 v, \zeta_3 w)$ for $\sigma = (\zeta_1, \zeta_2, \zeta_3) \in G$. Let $\chi : \mathbb{Q}[G] \rightarrow \overline{\mathbb{Q}}$ be the homomorphism of \mathbb{Q} -algebra given by $\chi(\zeta_1, \zeta_2, \zeta_3) = \zeta_1^{N(a-q)} \zeta_2^{N(b-q)} \zeta_3^{Nq}$. Let K be the coimage of χ , and $e \in \mathbb{Q}[G]$ the idempotent corresponding to $\mathbb{Q}[G] \rightarrow K$ (i.e. $e^2 = e$, $e\mathbb{Q}[G] = K$). Let D be a

union of curves in S defined by $(u^N - 1)(v^N - 1)w = 0$. The group G acts on D . We consider an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & eH_2(S, \mathbb{Q})/H_2(D) & \longrightarrow & eH_2(S, D; \mathbb{Q}) & \longrightarrow & eH_1(D, \mathbb{Q}) \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ & & H & & M & & \end{array} \quad (4.5)$$

of mixed de Rham-Hodge structures. Note that H is a de Rham-Hodge structure of type $(0, -2), (-1, -1), (-2, 0)$.

Lemma 4.1. $\dim_K eH_1(D) = 1$, $\dim_K eH_2(D) = 0$ and $\dim_K eH = \dim_K eH_2(S) = 1$. Moreover $eH_1(D, \mathbb{Q})$ is a de Rham-Hodge structure of type $(0, 0)$.

Proof. The former statement is an easy exercise. To see the latter, let $\tilde{D} \rightarrow D$ be the normalization and put $\mathbb{Q}_E := \text{Coker}(\mathbb{Q}_D \rightarrow \mathbb{Q}_{\tilde{D}})$. Then there is the exact sequence

$$H^0(\tilde{D}) \rightarrow H^0(\mathbb{Q}_E) \rightarrow H^1(D) \rightarrow H^1(\tilde{D}) \rightarrow 0$$

and it remains true after applying e . Since $q \notin \mathbb{Z}$ one has $eH^1(\tilde{D}) = eH^1(\{u^N + v^N - 1 = w = 0\}) = 0$ and the assertion follows. \square

The cycle Δ given in (4.3) defines a homology cycle in $H_2^B(S, D; \mathbb{Z})$. Let $\delta := e(\partial\Delta) \in eH_1^B(D, \mathbb{Q})$. The exact sequence (4.5) gives rise to the connecting homomorphism

$$\rho : eH_1^B(D, \mathbb{Q}) \cong K \rightarrow \text{Ext}^1(\mathbb{Q}, H) \quad (4.6)$$

to the Yoneda extension group of mixed de Rham-Hodge structures. Denote by $H^\chi = (H_B^\chi, H_{\text{dR}}^\chi, F^\bullet, \iota)$ the χ -part, i.e. the subspace on which each $\sigma \in G$ acts by multiplication by $\chi(\sigma)$. Let i_η be the composition of the following maps

$$\begin{aligned} \text{Ext}^1(\mathbb{Q}, H) &\xrightarrow{\cong} H_{\text{dR}, \mathbb{C}}/F^0 H_{\text{dR}} + \iota(H_B) \quad (H_{\text{dR}, \mathbb{C}} := \mathbb{C} \otimes_{\overline{\mathbb{Q}}} H_{\text{dR}}) \\ &\rightarrow H_{\text{dR}, \mathbb{C}}^\chi/F^0 H_{\text{dR}}^\chi + \iota(H_B^\chi) \quad (\text{projection}) \\ &\xrightarrow{\cong} \mathbb{C}/\text{Per}(H^\chi)^{\otimes -1} \text{ or } \mathbb{C}/(\text{Per}(H^\chi)^{\otimes -1} + \overline{\mathbb{Q}}) \end{aligned}$$

where the last isomorphism is induced from the dual basis $\eta^* \in H_{\text{dR}}^\chi \cong \text{Hom}(\overline{\mathbb{Q}}\eta, \overline{\mathbb{Q}})$ (cf. Lemma 4.1 (1)), and the choice of $\mathbb{C}/\text{Per}(H^\chi)^{\otimes -1}$ or $\mathbb{C}/(\text{Per}(H^\chi)^{\otimes -1} + \overline{\mathbb{Q}})$ depends on $F^0 H_{\text{dR}}^\chi = 0$ or not.

Lemma 4.2. Let $\rho^\chi := i_\eta \circ \rho$. There is a constant $c \in \overline{\mathbb{Q}}$ such that

$$\int_{\Delta} u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} du dv = \pm \rho^\chi(\delta) + c \quad (4.7)$$

in $\mathbb{C}/\text{Per}(H^\chi)^{\otimes -1}$ or $\mathbb{C}/(\text{Per}(H^\chi)^{\otimes -1} + \overline{\mathbb{Q}})$ (in the latter case, the constant c is annihilated).

Proof. By the exact sequence

$$0 \rightarrow eH_{\text{dR}}^1(D) \xrightarrow{h} eH_{\text{dR}}^2(S, D) \rightarrow eH_{\text{dR}}^2(S) \rightarrow eH_{\text{dR}}^2(D) = 0$$

there is a lifting $\tilde{\eta} \in eH_{\text{dR}}^2(S, D)^\chi$ of $\eta \in eH_{\text{dR}}^2(S)^\chi$. As is easily shown from definition (cf. [4] Prop.7.1),

$$\rho^\chi(\delta) = \pm \langle e\Delta, \tilde{\eta} \rangle + c_0, \quad \exists c_0 \in \overline{\mathbb{Q}}$$

where the right pairing denotes the pairing on $H_2^B(S, D) \otimes H_{\text{dR}}^2(S, D)$. The constant c_0 depends on the choice of the lifting $\tilde{\eta}$. If $F^0 H_{\text{dR}}^\chi = 0$ ($\Leftrightarrow \eta \in F^1 H_{\text{dR}}^2(S)$), one can choose a unique lifting $\tilde{\eta} \in eH_{\text{dR}}^2(S, D)^\chi \cap F^1$ and then $c_0 = 0$.

On the other hand let $\tilde{\eta}' = (\eta, 0, 0) \in eH_{\text{dR}}^2(S, D)$ be a lifting by the notation in [3] (7.2). Then

$$\int_{\Delta} u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} dudv = \langle \Delta, \tilde{\eta}' \rangle = \langle e\Delta, \tilde{\eta}' \rangle.$$

Letting $\xi \in eH_{\text{dR}}^1(D)$ such that $\tilde{\eta}' - \tilde{\eta} = h(\xi)$ we thus have

$$\begin{aligned} \int_{\Delta} u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} dudv \pm \rho^\chi(\delta) &= \langle e\Delta, \tilde{\eta}' - \tilde{\eta} \rangle \pm c_0 \\ &= \langle e\partial\Delta, \xi \rangle \pm c_0 \\ &\in \text{Im}(eH_1^B(D) \otimes eH_{\text{dR}}^1(D)) + \overline{\mathbb{Q}} = \overline{\mathbb{Q}}. \end{aligned}$$

□

The connecting map (4.6) sits into a diagram

$$\begin{array}{ccccc} H_{\mathcal{M}, D}^3(S, \mathbb{Q}(2)) & \longrightarrow & H_{\mathcal{M}}^3(S, \mathbb{Q}(2)) & & \\ \text{reg}_D \downarrow & & \text{reg}_S \downarrow & & \\ H_1^B(D, \mathbb{Q}) & \xrightarrow{\rho} & \text{Ext}^1(\mathbb{Q}, H) & \xrightarrow{i_\eta} & \mathbb{C}/\text{Per}(H^\chi)^{\otimes -1} + \overline{\mathbb{Q}} \end{array}$$

which is commutative up to sign, where reg_D and reg_S are the regulator maps. Thus the same argument as in the proof of Lemma 3.2 works and we have the following.

Lemma 4.3. *There is a cycle $z \in H_{\mathcal{M}}^3(S, \mathbb{Q}(2))$ such that*

$$\int_{\Delta} u^{N(a-q)-1} v^{N(b-q)-1} w^{Nq-N} dudv = \pm i_\eta \circ \text{reg}_S(z) \in \mathbb{C}/\text{Per}(H^\chi)^{\otimes -1} + \overline{\mathbb{Q}}. \quad (4.8)$$

In particular, if $H = eH_2(S)$ has the Hodge type $(-1, -1)$, it follows from (4.4) and the fact $\text{Per}(H^\chi)^{\otimes -1} = 2\pi i \overline{\mathbb{Q}}$ that we have

$$B(a, b)_3 F_2 \left(\begin{matrix} a, q, b \\ a + b, q + 1 \end{matrix}; 1 \right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log(\overline{\mathbb{Q}}^\times).$$

Lemma 4.4. *$H = eH_2(S, \mathbb{Q})$ has the Hodge type $(-1, -1)$ if and only if*

$$\{sq\} + \{s(-q + a)\} + \{s(-q + b)\} + \{s(q - a - b)\} = 2$$

holds for all $s \in \hat{\mathbb{Z}}^\times$.

Proof. $eH_2(S, \mathbb{Q})$ has the Hodge type $(-1, -1)$ if and only if $eH^2(S, \mathbb{Q})$ has the Hodge type $(1, 1)$. As is well-known, the de Rham cohomology $eH_{\text{dR}}^2(S)$ is generated by rational 2-forms

$$\eta_s := u^{sN(a-q)-1} v^{sN(b-q)-1} w^{sNq} dudv, \quad s \in (\mathbb{Z}/N\mathbb{Z})^\times$$

and η_s belongs to the Hodge $(p_s, 2 - p_s)$ -component where $p_s := \{sq\} + \{s(-q + a)\} + \{s(-q + b)\} + \{s(q - a - b)\} - 1$. Now the assertion is immediate. □

Theorem 2.1 is immediate from Lemmas 4.3 and 4.4.

5. OPEN PROBLEMS

Problem 5.1. *Give an explicit description of (2.2) in terms of logarithmic function.*

We obtained explicit descriptions in some cases of $(a, b, q) \in S$ (e.g. the example in §1) by developing the technique in §3 or §4. However there still remains technical difficulties arising from algebraic geometry to compute all cases. We will discuss this issue in detail in a forthcoming paper.

We propose a conjecture of a necessary and sufficient condition for that ${}_3F_2$ is written in terms of log:

Conjecture 5.2. *Let $a, b, q \in \mathbb{Q}$ be non-integers such that none of $q-a, q-b, q-a-b$ are integers. Then*

$$B(a, b) {}_3F_2 \left(\begin{matrix} a, b, q \\ a+b, q+1 \end{matrix}; 1 \right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times \quad (5.1)$$

if and only if

$$\{sq\} + \{s(-q+a)\} + \{s(-q+b)\} + \{s(q-a-b)\} = 2, \quad \forall s \in \hat{\mathbb{Z}}^\times. \quad (5.2)$$

As we have seen before, (5.2) is a necessary and sufficient condition for that $eH^{(l)}$ in §3 or eH in §4 is a Tate motive. Otherwise they contains no Tate motives, and the complex number (5.1) turns out to be Beilinson's regulator of an extension of such a motive. Then it is quite weird if it were written only in terms of logarithmic function.

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